

Vector bundles on p -adic curves and parallel transport II

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1 Introduction

In [DW1] we established a partial p -adic analogue of the Narasimhan–Seshadri correspondence between vector bundles and representations of the fundamental group. In [Fa] Faltings has even constructed a p -adic analogue of Simpson’s theory of Higgs bundles. In his theory, p -adic representations of the fundamental group of a p -adic curve give rise to semistable p -adic Higgs bundles of slope zero. Faltings asks whether this condition is also sufficient for a p -adic Higgs bundle to come from a p -adic representation.

Let X be a smooth projective curve over $\overline{\mathbb{Q}}_p$. In [DW1] we defined a category \mathfrak{B}_X^{ps} of vector bundles on $X \otimes \mathbb{C}_p$ with “potentially strongly semistable reduction of degree zero”. For these we constructed a theory of parallel transport along (homotopy classes of) étale paths and in particular representations of the fundamental group of X . Every vector bundle in \mathfrak{B}_X^{ps} is semistable of slope zero on $X_{\mathbb{C}_p}$. Prompted by Faltings’ question we wondered about the converse. In [DW2] we showed that \mathfrak{B}_X^{ps} equipped with the natural fibre functor in a point of $X_{\mathbb{C}_p}$ is a neutral Tannakian category just like the full category of semistable vector bundles of slope zero on $X_{\mathbb{C}_p}$.

Let $f : Y \rightarrow X$ be a possibly ramified finite covering of smooth projective curves over $\overline{\mathbb{Q}}_p$. It is well known that a vector bundle E on $X_{\mathbb{C}_p}$ is semistable if and only if f^*E is semistable on $Y_{\mathbb{C}_p}$. If f is unramified then it is easy to see

that E is in \mathfrak{B}_X^{ps} if and only if f^*E is in \mathfrak{B}_Y^{ps} . However in the ramified case we do not know if this is true.

In the present paper we define for every $\mu \in \mathbb{Q}$ a category \mathfrak{B}_X^μ of vector bundles of slope μ on $X_{\mathbb{C}_p}$ with “potentially strongly semistable reduction”. The bundles in \mathfrak{B}_X^μ are semistable of slope μ on $X_{\mathbb{C}_p}$ and \mathfrak{B}_X^0 contains \mathfrak{B}_X^{ps} . We show that a bundle E is in \mathfrak{B}_X^0 if and only if f^*E is in \mathfrak{B}_Y^0 even if f is ramified, and similarly for non-zero slope. On the other hand we extend our theory of parallel transport from \mathfrak{B}_X^{ps} to \mathfrak{B}_X^0 . Since \mathfrak{B}_X^0 is again a neutral Tannakian category it may be reasonable to conjecture that every semistable vector bundle of slope zero on $X_{\mathbb{C}_p}$ lies in \mathfrak{B}_X^0 i.e. has potentially strongly semistable reduction.

The main difference in the definitions of \mathfrak{B}_X^{ps} and \mathfrak{B}_X^0 lies in the meaning of the word “potential”. For E to be in \mathfrak{B}_X^{ps} we allowed only pullbacks by finite étale coverings α of X before α^*E was supposed to have “strongly semistable reduction of degree zero”. For bundles E in \mathfrak{B}_X^0 on the other hand we allow arbitrary ramified coverings α before α^*E is supposed to have “strongly semistable reduction”. The problem in defining the parallel transport for E in \mathfrak{B}_X^0 comes down to this: Using the theory of [DW1] we can define a parallel transport between the fibres of α^*E for all étale paths on $Y_{\mathbb{C}_p}$. It is easy to see that this parallel transport descends to a parallel transport for E along the étale paths in $U \subset X$, where U is the complement of the ramification points of α . In particular, for $x_0 \in U(\mathbb{C}_p)$ one obtains a representation $\rho : \pi_1(U, x_0) \rightarrow \mathrm{GL}(E_{x_0})$. The main point is: This representation has no monodromy at the ramification points i.e. that it factors over $\pi_1(X, x_0)$. We were unable to prove this algebraically. Instead our proof uses Grothendieck’s comparison theorem between algebraic and topological fundamental groups and some considerations on Riemann surfaces.

We were motivated by the following heuristic argument: A suitable small deformation of α and Y would not affect the reduction of α^*E but replace the ramification locus of α on X by a disjoint ramification locus. Thus we would have $U \cup U' = X$ for the new complement U' . By general arguments from [DW1] and in particular proposition 34 the parallel transports on U and U' have to be compatible and glue to a parallel transport on X . In particular there would be no monodromy.

At the end of [Fa], Faltings remarks that if the pullback with respect to a possibly ramified covering of one of his generalized representations ρ of the fundamental group is a true representation, then ρ is a true representation as

well. This fact is compatible with a positive answer to his above question.

In the final section we extend another part of [NS], dealing with vector bundles of arbitrary slope, to the p -adic context. If X has non-zero genus then for $r \geq 1$ there is a canonical central extension

$$1 \longrightarrow \mu_r \longrightarrow \Gamma_r \longrightarrow \pi_1(X, x_0) \longrightarrow 1$$

where μ_r denotes the group of r -th roots of unity in \mathbb{C}_p . For any vector bundle E of slope $\mu = d/r$ in \mathfrak{B}_X^μ we construct a representation

$$\rho_{E,x_0} : \Gamma_r \longrightarrow \mathrm{GL}(E_{x_0}).$$

For this we use a similar topological argument as the one employed for proving the absence of monodromy above. If E has degree zero then ρ_{E,x_0} factors over $\pi_1(X, x_0)$ and is the representation defined previously for bundles in \mathfrak{B}_X^0 .

The present paper originated from a question by Luis Álvarez-Cónsul to the first author whether the Narasimhan–Seshadri correspondence for vector bundles of non-zero slopes had a p -adic analogue. We are very grateful to him for this insight. We would also like to thank Oscar García-Prada very much for several explanations of the classical case.

2 Vector bundles with strongly semistable reduction

Recall that for a vector bundle E on a smooth, projective and connected curve C over a field k the slope is defined by $\mu(E) = \deg(E)/\mathrm{rk}(E)$. The bundle E is called semistable (respectively stable), if for all proper non-zero subbundles F of E the inequality $\mu(F) \leq \mu(E)$ (respectively $\mu(F) < \mu(E)$) holds.

If $\mathrm{char}(k) = 0$, then pullback by any surjective morphism of smooth connected projective curves preserves semistability of vector bundles. However in the case $\mathrm{char}(k) = p$, there exist vector bundles which are destabilized by the Frobenius map.

Assume that $\mathrm{char}(k) = p$, and let $F : C \rightarrow C$ be the absolute Frobenius morphism, defined by the p -power map on the structure sheaf. In this case, a

vector bundle E on C is called strongly semistable, if $F^{n*}E$ is semistable on C for all $n \geq 1$.

Strong semistability in characteristic p is preserved by tensor products and by pullback with respect to k -morphisms between smooth connected projective curves, see [Mi], § 5.

Now consider a one-dimensional proper scheme Z over a field k of characteristic p . By C_1, \dots, C_r we denote the irreducible components of Z endowed with their reduced induced structures. Let \tilde{C}_i be the normalization of C_i , and write $\alpha_i : \tilde{C}_i \rightarrow C_i \rightarrow Z$ for the canonical morphism. Note that the \tilde{C}_i are either points or smooth, projective and connected curves over k .

Definition 1 *Let E be a vector bundle on Z . Then E is called strongly semistable, if one of the following equivalent conditions hold:*

- i) *For all one-dimensional irreducible components C_i of Z , the pullback α_i^*E is strongly semistable on \tilde{C}_i .*
- ii) *For all smooth, projective and connected curves C over k and all k -morphisms $\alpha : C \rightarrow Z$, the pullback α^*E is semistable on C .*

The equivalence of the two conditions can be shown as in [DW2], 12.2.4. Note however, that the slopes of the pullbacks α_i^*E may depend on i .

If $f : Z' \rightarrow Z$ is a k -morphism between one-dimensional proper k -schemes and E is a strongly semistable vector bundle on Z , then f^*E is a strongly semistable vector bundle on Z' (use ii) above).

By $\overline{\mathbb{Z}}_p$ and \mathfrak{o} we denote the rings of integers in $\overline{\mathbb{Q}}_p$ and \mathbb{C}_p , and by $k = \overline{\mathbb{F}}_p$ the common residue field of $\overline{\mathbb{Z}}_p$ and \mathfrak{o} .

Let X be a smooth, projective and connected curve over $\overline{\mathbb{Q}}_p$, and let $X_{\mathbb{C}_p}$ be its base-change to \mathbb{C}_p . We call any finitely presented, proper and flat scheme \mathfrak{X} over $\overline{\mathbb{Z}}_p$ with generic fibre X a model of X . Set $\mathfrak{X}_{\mathfrak{o}} = \mathfrak{X} \otimes_{\overline{\mathbb{Z}}_p} \mathfrak{o}$ and $\mathfrak{X}_k = \mathfrak{X} \otimes_{\overline{\mathbb{Z}}_p} k$.

Definition 2 *We say that a vector bundle E of arbitrary slope on $X_{\mathbb{C}_p}$ has strongly semistable reduction if E is isomorphic to the generic fibre of a vector*

bundle \mathcal{E} on \mathfrak{X}_0 for some model \mathfrak{X} of X , such that the special fibre \mathcal{E}_k is a strongly semistable vector bundle on the one-dimensional proper k -scheme \mathfrak{X}_k in the sense of definition 1.

We say that E has potentially strongly semistable reduction, if there is a finite morphism $\alpha : Y \rightarrow X$ of connected smooth projective curves over $\overline{\mathbb{Q}}_p$ such that $\alpha_{\mathbb{C}_p}^* E$ has strongly semistable reduction on $Y_{\mathbb{C}_p}$.

For all slopes $\mu \in \mathbb{Q}$ we denote by \mathfrak{B}_X^μ or simply \mathfrak{B}^μ the full subcategory of all vector bundles on $X_{\mathbb{C}_p}$ of slope μ which have potentially strongly semistable reduction.

In [DW1] and [DW2] we considered the category \mathfrak{B}_X^s of vector bundles on $X_{\mathbb{C}_p}$ with strongly semistable reduction of degree zero. Here E has strongly semistable reduction of degree zero if there exists a model \mathcal{E} on some \mathfrak{X}_0 as above, such that the pull-back of \mathcal{E}_k to each normalized irreducible component of \mathfrak{X}_k is strongly semistable of degree zero. In particular this implies that E has degree zero. Note that a bundle of degree zero which has strongly semistable reduction in the more general sense of definition 2 may have different degrees on the components of the special fibre.

Besides we defined the category \mathfrak{B}_X^{ps} as the category of vector bundles on $X_{\mathbb{C}_p}$ for which there exists a finite étale covering $\beta : Y \rightarrow X$ by a connected curve Y over $\overline{\mathbb{Q}}_p$ such that $\beta_{\mathbb{C}_p}^* E$ lies in \mathfrak{B}_Y^s . In the definition of \mathfrak{B}^μ on the other hand we allow ramified coverings. There are the following inclusions of categories

$$\mathfrak{B}_X^s \subset \mathfrak{B}_X^{ps} \subset \mathfrak{B}_X^0 .$$

The following result provides a link between our old and our new categories.

Theorem 3 *i) Let E be a vector bundle of slope μ on $X_{\mathbb{C}_p}$. Then E has potentially strongly semistable reduction, i.e. E lies in \mathfrak{B}_X^μ , if and only if there exists a finite morphism $\alpha : Y \rightarrow X$ of smooth, projective, connected curves over $\overline{\mathbb{Q}}_p$ and a line bundle L on $Y_{\mathbb{C}_p}$ such that the bundle $\alpha_{\mathbb{C}_p}^* E \otimes L$ lies in the category \mathfrak{B}_Y^s .*

ii) We can replace the category \mathfrak{B}^s in i) by the category \mathfrak{B}^{ps} , i.e. E lies in \mathfrak{B}_X^μ if and only if there exists a finite morphism $\alpha : Y \rightarrow X$ of smooth, projective,

connected curves over $\overline{\mathbb{Q}}_p$ and a line bundle L on $Y_{\mathbb{C}_p}$ such that $\alpha_{\mathbb{C}_p}^* E \otimes L$ lies in \mathfrak{B}_Y^{ps} . If E satisfies this condition, then every line bundle L on $Y_{\mathbb{C}_p}$ satisfying $\deg L = -\mu \deg \alpha$ has the property that $\alpha_{\mathbb{C}_p}^* E \otimes L$ lies in \mathfrak{B}_Y^{ps} .

iii) If E_1, \dots, E_r are finitely many vector bundles in \mathfrak{B}_X^μ , then there is one covering Y working for all of them. To be precise, there exists a finite dominant morphism $\alpha : Y \rightarrow X$ of smooth, projective, connected curves over $\overline{\mathbb{Q}}_p$ and line bundles L_1, \dots, L_r on $Y_{\mathbb{C}_p}$ such that the bundles $\alpha_{\mathbb{C}_p}^* E_i \otimes L_i$ lie in \mathfrak{B}_Y^s . Moreover, for any line bundle L on $Y_{\mathbb{C}_p}$ with $\deg L = -\mu \deg \alpha$ all bundles $\alpha_{\mathbb{C}_p}^* E_i \otimes L$ lie in \mathfrak{B}_Y^{ps} .

Proof Let E be a vector bundle of slope μ on $X_{\mathbb{C}_p}$.

i) Let us first assume that there is a finite covering $\alpha : Y \rightarrow X$ by a smooth, connected curve Y and a line bundle L on $Y_{\mathbb{C}_p}$ such the bundle $F = \alpha_{\mathbb{C}_p}^* E \otimes L$ lies in \mathfrak{B}_Y^s . This means that there exists a model \mathcal{Y} of Y and a model \mathcal{F} of F on $\mathcal{Y}_\circ = \mathcal{Y} \otimes \mathfrak{o}$ such that the pullback of the special fibre \mathcal{F}_k to all normalized irreducible components \tilde{C}_i of \mathcal{Y}_k is strongly semistable of degree zero. By [DW1], theorem 5 there exists a model \mathcal{Y}' of Y dominating \mathcal{Y} such that L can be extended to a line bundle \mathcal{L} on \mathcal{Y}'_\circ . If $\pi : \mathcal{Y}' \rightarrow \mathcal{Y}$ denotes the corresponding morphism, then the special fibre $(\pi^* \mathcal{F})_k = \pi_k^* \mathcal{F}_k$ of $\pi^* \mathcal{F}$ is the pullback of a vector bundle which is strongly semistable of degree zero on all normalized irreducible components of \mathcal{Y} . Hence it is also semistable of degree zero on all normalized irreducible components of \mathcal{Y}' .

Therefore we can substitute \mathcal{Y} by \mathcal{Y}' and \mathcal{F} by $\pi^* \mathcal{F}$ and assume that L is the generic fibre of a line bundle \mathcal{L} on \mathcal{Y}_\circ . The pullback of its special fibre \mathcal{L}_k to any normalized irreducible component \tilde{C}_i of \mathcal{Y}_k is strongly semistable since it is a line bundle. Hence the pullback of $(\mathcal{F} \otimes \mathcal{L}^{-1})_k = \mathcal{F}_k \otimes \mathcal{L}_k^{-1}$ to \tilde{C}_i is strongly semistable. Since $\mathcal{F} \otimes \mathcal{L}^{-1}$ is a vector bundle on \mathcal{Y}_\circ with generic fibre $\alpha_{\mathbb{C}_p}^* E$, we find that $\alpha_{\mathbb{C}_p}^* E$ has strongly semistable reduction in the sense of definition 2. Therefore E lies indeed in \mathfrak{B}^μ .

Now let us conversely assume that E lies in \mathfrak{B}_X^μ . We denote the degree of E by d , and its rank by r . Hence $\mu = d/r$. By assumption, there exists a finite morphism $\alpha : Y \rightarrow X$, a model \mathcal{Y} of Y and a vector bundle \mathcal{F} on \mathcal{Y}_\circ with generic fibre $\alpha_{\mathbb{C}_p}^* E$ such that its special fibre \mathcal{F}_k is strongly semistable in the sense of definition 1. Since \mathcal{Y} is finitely presented over $\overline{\mathbb{Z}}_p$, it descends to a

scheme over the ring of integers in a finite extension K_0 of \mathbb{Q}_p . Let K be the maximal unramified extension of K_0 , and let \mathfrak{o}_K denote its ring of integers. It is a discrete valuation ring with residue field k . Then \mathcal{Y} descends to a flat, proper and finitely presented \mathfrak{o}_K -scheme $\mathcal{Y}_{\mathfrak{o}_K}$. Its special fibre is $(\mathcal{Y}_{\mathfrak{o}_K})_k = \mathcal{Y}_k$. Note that $\mathcal{Y}_{\mathfrak{o}_K}$ is irreducible and reduced by [Liu], Proposition 4.3.8. By lemma 4 proven below there exists an irreducible, semistable, proper \mathfrak{o}_K -scheme \mathcal{Z} with smooth generic fibre and a morphism of \mathfrak{o}_K -schemes

$$\beta : \mathcal{Z} \rightarrow \mathcal{Y}_{\mathfrak{o}_K}$$

with finite generic fibre β_K such that for the irreducible components D_1, \dots, D_t of the special fibre \mathcal{Z}_k the following condition holds: If β_k maps D_i surjectively to an irreducible component C_i of \mathcal{Y}_k (i.e. not to a point), then the degree k_i of the finite morphism

$$\beta_k|_{D_i} : D_i \rightarrow C_i,$$

where the components are endowed with their reduced structures, is a multiple of r . We denote by \tilde{D}_i the normalization of D_i . If $\beta_k(D_i) = C_i$, then $\beta_k|_{D_i} : D_i \rightarrow C_i$ induces a morphism $\tilde{D}_i \rightarrow \tilde{C}_i$ of the same degree k_i .

Now we look at the bundle $\beta_k^* \mathcal{F}_k$. If β_k maps the component D_i of \mathcal{Z}_k to a point, then the pullback of $\beta_k^* \mathcal{F}_k$ to D_i (and hence to \tilde{D}_i) is trivial, hence strongly semistable of degree zero.

On the other hand, if $\beta_k(D_i)$ is equal to the component C_i , then we denote by $(\beta_k^* \mathcal{F}_k)|_{\tilde{D}_i}$ the pullback of $\beta_k^* \mathcal{F}_k$ to \tilde{D}_i . The bundle $(\beta_k^* \mathcal{F}_k)|_{\tilde{D}_i}$ is equal to the pullback of $\mathcal{F}_k|_{\tilde{C}_i}$ via the map $\tilde{D}_i \rightarrow \tilde{C}_i$ induced by β_k . Since $\mathcal{F}_k|_{\tilde{C}_i}$ is strongly semistable, say of degree e_i , its pullback to \tilde{D}_i is strongly semistable of degree $e_i k_i$. Hence the degree of $(\beta_k^* \mathcal{F}_k)|_{\tilde{D}_i}$ is a multiple of r .

Now we want to construct for every irreducible component D_i of \mathcal{Z}_k a line bundle $\mathcal{L}(i)$ on \mathcal{Z} such that the special fibre $\mathcal{L}(i)_k$ satisfies

$$\deg(\mathcal{L}(i)_k|_{\tilde{D}_j}) = \delta_{ij},$$

where δ_{ij} is the Kronecker delta.

Since \mathcal{Z} is semistable, the singular locus S in the special fibre \mathcal{Z}_k consists of finitely many closed points. Let $\mathcal{Z}' = \mathcal{Z} \setminus S$ be the complement. Then \mathcal{Z}' is a smooth \mathfrak{o}_K -scheme such that the special fibre \mathcal{Z}'_k is dense in \mathcal{Z}_k . Hence every

irreducible component D_i of \mathcal{Z}_k contains a closed point $P_i \in \mathcal{Z}'_k$. Since k is algebraically closed, we have $P_i \in \mathcal{Z}'_k(k)$. Note that the special fibre map

$$\mathcal{Z}'(\mathfrak{o}_K) \rightarrow \mathcal{Z}'_k(k)$$

is surjective, see e.g. [BLR], 2.3, Proposition 5. Hence we find some $\Delta_i \in \mathcal{Z}'(\mathfrak{o}_K)$ with special fibre P_i .

We identify Δ_i with its image in \mathcal{Z}' , which we regard as a Weil divisor. Since \mathcal{Z}' is regular, Δ_i gives rise to a line bundle $\mathcal{O}(\Delta_i)$ on \mathcal{Z}' . It is trivial on the open subscheme $\mathcal{Z}' \setminus \Delta_i$ of \mathcal{Z}' . Hence we can glue it with the trivial bundle on $\mathcal{Z} \setminus \Delta_i$ and obtain a line bundle $\mathcal{L}(i)$ on \mathcal{Z} . By construction we have

$$\deg(\mathcal{L}(i)_k|_{D_j}) = 0, \text{ if } j \neq i \quad \text{and} \quad \deg(\mathcal{L}(i)_k|_{D_i}) = 1.$$

Therefore the line bundle $\mathcal{L}(i)$ on \mathcal{Z} satisfies indeed

$$\deg(\mathcal{L}(i)_k|_{\tilde{D}_j}) = \delta_{ij}.$$

Now we consider again the vector bundle $\beta_k^* \mathcal{F}_k$ on the special fibre \mathcal{Z}_k . We have shown that for all components D_i the vector bundle $(\beta_k^* \mathcal{F}_k)|_{\tilde{D}_i}$ is strongly semistable on \tilde{D}_i such that $d_i = \deg(\beta_k^* \mathcal{F}_k)|_{\tilde{D}_i}$ is a multiple of r . Let \mathcal{L}^* denote the following line bundle on \mathcal{Z} :

$$\mathcal{L}^* = \mathcal{L}(1)^{d_1/r} \otimes \dots \otimes \mathcal{L}(t)^{d_t/r}.$$

We consider the base changes $\mathcal{Z}_{\overline{\mathbb{Z}}_p} = \mathcal{Z} \otimes_{\mathfrak{o}_K} \overline{\mathbb{Z}}_p$ and $\mathcal{Z}_{\mathfrak{o}} = \mathcal{Z} \otimes_{\mathfrak{o}_K} \mathfrak{o}$, and denote by

$$\beta_{\overline{\mathbb{Z}}_p} : \mathcal{Z}_{\overline{\mathbb{Z}}_p} \rightarrow \mathcal{Y}_{\mathfrak{o}_K} \otimes_{\mathfrak{o}_K} \overline{\mathbb{Z}}_p = \mathcal{Y}$$

the base change map. Then its generic fibre

$$\beta_{\overline{\mathbb{Z}}_p} \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p : Z = \mathcal{Z}_{\overline{\mathbb{Q}}_p} \rightarrow \mathcal{Y}_{\overline{\mathbb{Q}}_p} = Y$$

is a finite morphism. Composing it with $\alpha : Y \rightarrow X$, we get a finite morphism

$$\gamma = \alpha \circ (\beta_{\overline{\mathbb{Z}}_p} \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p) : Z \rightarrow X$$

of smooth, projective, connected curves over $\overline{\mathbb{Q}}_p$. The vector bundle $\beta_{\mathfrak{o}}^* \mathcal{F}$ on $\mathcal{Z}_{\mathfrak{o}}$ is a model of the vector bundle $\gamma_{\mathbb{C}_p}^* E$ on the generic fibre. Now let $\mathcal{L} = \mathcal{L}^* \otimes_{\mathfrak{o}_K} \mathfrak{o}$

be the base change of \mathcal{L}^* , and let $L = \mathcal{L} \otimes_{\mathfrak{o}} \mathbb{C}_p$ denote its generic fibre. Then the vector bundle $\beta_{\mathfrak{o}}^* \mathcal{F} \otimes \mathcal{L}^{-1}$ on $\mathcal{Z}_{\mathfrak{o}}$ has generic fibre $\gamma_{\mathbb{C}_p}^* E \otimes L^{-1}$. Since the residue fields of \mathfrak{o}_K and \mathfrak{o} coincide, the special fibre of $\beta_{\mathfrak{o}}^* \mathcal{F} \otimes \mathcal{L}^{-1}$ is equal to $(\beta_k^* \mathcal{F}_k) \otimes \mathcal{L}_k^{*-1}$. Pulling back to the normalized irreducible component \tilde{D}_i of \mathcal{Z}_k we get the vector bundle

$$(\beta_k^* \mathcal{F}_k)|_{\tilde{D}_i} \otimes \mathcal{L}_k^{*-1}|_{\tilde{D}_i}.$$

We know that $(\beta_k^* \mathcal{F}_k)|_{\tilde{D}_i}$ is strongly semistable of degree d_i on \tilde{D}_i . As a line bundle, $\mathcal{L}_k^{*-1}|_{\tilde{D}_i}$ is also strongly semistable on \tilde{D}_i . By construction, its degree is equal to $-d_i/r$. Therefore $(\beta_k^* \mathcal{F}_k)|_{\tilde{D}_i} \otimes \mathcal{L}_k^{*-1}|_{\tilde{D}_i}$ is strongly semistable of degree 0. Hence $\gamma_{\mathbb{C}_p}^* E$ lies indeed in \mathfrak{B}_Z^s .

ii) Assume that there exists a finite covering $\alpha : Y \rightarrow X$ by a smooth, connected curve Y and a line bundle L on $Y_{\mathbb{C}_p}$ such the bundle $F = \alpha_{\mathbb{C}_p}^* E \otimes L$ lies in \mathfrak{B}_Y^{ps} . This means that there exists a finite étale covering $\beta : Y' \rightarrow Y$ by a projective, connected curve Y' such that $\beta_{\mathbb{C}_p}^* (\alpha_{\mathbb{C}_p}^* E \otimes L)$ lies in $\mathfrak{B}_{Y'}^s$. Replacing α by $\alpha \circ \beta$ and L by $\beta_{\mathbb{C}_p}^* L$, we are in the situation of i). Therefore E lies in \mathfrak{B}_X^{μ} .

Now assume additionally that M is any line bundle on $Y_{\mathbb{C}_p}$ satisfying $\deg M = -\mu \deg \alpha$. Then $L^{-1} \otimes M$ is a line bundle of degree zero on $Y_{\mathbb{C}_p}$. By [DW1], Theorem 12, $L^{-1} \otimes M$ lies in \mathfrak{B}_Y^{ps} . Since \mathfrak{B}_Y^{ps} is stable under tensor product by [DW1], proposition 9, we find that $\alpha_{\mathbb{C}_p}^* E \otimes M$ lies indeed in \mathfrak{B}_Y^{ps} .

iii) Let E_1, \dots, E_r be vector bundles in \mathfrak{B}_X^{μ} . Then by i) there exist finite coverings $\alpha_i : Y_i \rightarrow X$ and line bundles L_i on $Y_{i\mathbb{C}_p}$ such that $\alpha_{i\mathbb{C}_p}^* E \otimes L_i$ lies in $\mathfrak{B}_{Y_i}^s$ for all i . There is a finite covering $\alpha : Y \rightarrow X$ by a smooth, projective, connected curve Y factoring over all α_i . Since pullback via $Y \rightarrow Y_i$ maps $\mathfrak{B}_{Y_i}^s$ to \mathfrak{B}_Y^s by [DW1], Proposition 9, the first claim follows. The second is a consequence of ii). \square

Lemma 4 *Let R be a discrete valuation ring with quotient field K and residue field k , and let \mathcal{Y} be an irreducible, reduced, flat and proper scheme over R with one-dimensional fibres, such that the generic fibre is smooth. Then for every integer $r \geq 1$ there exists an irreducible semistable proper R -scheme \mathcal{Z} and an R -morphism $\beta : \mathcal{Z} \rightarrow \mathcal{Y}$ such that the following conditions hold:*

i) *The generic fibre \mathcal{Z}_K is smooth, and $\beta_K : \mathcal{Z}_K \rightarrow \mathcal{Y}_K$ is finite.*

ii) For any irreducible component D of \mathcal{Z}_k such that the special fibre β_k maps D surjectively to an irreducible component C of \mathcal{Y}_k (i.e. not to a point), the degree of the corresponding map of irreducible curves (endowed with their reduced structures) $\beta_k|_D : D \rightarrow C$ is a multiple of r .

Proof Let \mathcal{Y}' be the normalization of \mathcal{Y} in its function field. Then the corresponding morphism $\pi : \mathcal{Y}' \rightarrow \mathcal{Y}$ is finite and an isomorphism on the generic fibre. Since it suffices to show our claim for the normalization \mathcal{Y}' , we can assume that \mathcal{Y} is normal. Let C_1, \dots, C_t be the irreducible components of the special fibre \mathcal{Y}_k , and denote by η_i their generic points. For any i the local ring $\mathcal{O}_{\mathcal{Y}, \eta_i}$ is a discrete valuation ring in the function field $K(\mathcal{Y})$. The residue field of $\mathcal{O}_{\mathcal{Y}, \eta_i}$ is also the residue field of the local ring $\mathcal{O}_{\mathcal{Y}_k, \eta_i}$ in the special fibre. Now $\mathcal{O}_{\mathcal{Y}_k, \eta_i}$ is a noetherian local ring of dimension 0, hence it is Artinian and its residue field is simply the corresponding reduced ring $\mathcal{O}_{\mathcal{Y}_k, \eta_i}^{\text{red}}$. If we endow the component C_i with its induced reduced structure, we have $\mathcal{O}_{C_i, \eta_i} = \mathcal{O}_{\mathcal{Y}_k, \eta_i}^{\text{red}}$. Hence the residue field of $\mathcal{O}_{\mathcal{Y}, \eta_i}$ is a function field of dimension one over k . Therefore it is the quotient field of a factorial ring (even of a discrete valuation ring). In particular, there exist irreducible polynomials, e.g. Eisenstein polynomials, of arbitrary degree over the residue field of $\mathcal{O}_{\mathcal{Y}, \eta_i}$. Hence there exists a monic polynomial $f(T) \in \mathcal{O}_{\mathcal{Y}, \eta_i}[T]$ of degree r such that its reduction modulo the valuation ideal is irreducible over the residue field.

Then

$$B_i = \mathcal{O}_{\mathcal{Y}, \eta_i}[T]/(f(T))$$

is a discrete valuation ring dominating $\mathcal{O}_{\mathcal{Y}, \eta_i}$ such that the corresponding extension of residue fields has degree r , see [S] Ch. I, Proposition 15. Let L_i be its quotient field. Then L_i is an extension of $K(\mathcal{Y})$ of degree r , such that B_i is the only discrete valuation ring in L_i dominating $\mathcal{O}_{\mathcal{Y}, \eta_i}$.

Let L be the compositum of the extension fields L_i in some algebraic closure of $K(\mathcal{Y})$. Let \mathcal{Z} be the integral closure of the model \mathcal{Y} in L . Then there is a finite morphism $\beta : \mathcal{Z} \rightarrow \mathcal{Y}$. If D is an irreducible component of the special fibre \mathcal{Z}_k with generic point μ , then $\beta_k(D)$ is an irreducible component C_i of \mathcal{Y}_k . Hence $\mathcal{O}_{\mathcal{Z}, \mu}$ is a discrete valuation ring in L dominating $\mathcal{O}_{\mathcal{Y}, \eta_i}$. Its intersection with L_i is a discrete valuation ring in L_i dominating $\mathcal{O}_{\mathcal{Y}, \eta_i}$, hence it is equal to B_i . Therefore the degree of the extension of residue fields corresponding to $\mathcal{O}_{\mathcal{Y}, \eta_i} \subset \mathcal{O}_{\mathcal{Z}, \mu}$ is a multiple of r .

If we endow the component D of \mathcal{Z}_k with its induced reduced structure, the function field $K(D)$ of the curve D is equal to the local ring $\mathcal{O}_{D,\mu}$. We have seen above that $\mathcal{O}_{D,\mu}$ coincides with the reduced ring associated to $\mathcal{O}_{\mathcal{Z}_k,\mu} = \mathcal{O}_{\mathcal{Z},\mu} \otimes_R k$. Hence the residue field of $\mathcal{O}_{\mathcal{Z},\mu}$ is equal to $\mathcal{O}_{D,\mu} = K(D)$.

Therefore the degree of the map $\beta_k|_D : D \rightarrow C_i$ is just the residue degree of $\mathcal{O}_{\mathcal{Y},\eta_i} \subset \mathcal{O}_{\mathcal{Z},\mu}$ and hence a multiple of r .

To conclude the proof, we replace \mathcal{Z} by a semistable model \mathcal{Z}' of the generic fibre of \mathcal{Z} dominating \mathcal{Z} . \square

Corollary 5 *Let μ be any rational number.*

- i) *Every bundle in \mathfrak{B}_X^μ is semistable of slope μ .*
- ii) *If E is a vector bundle in the category \mathfrak{B}_X^μ and F is a vector bundle in the category \mathfrak{B}_X^ν , then $E \otimes F$ lies in the category $\mathfrak{B}_X^{\mu+\nu}$. Besides, the internal hom bundle $\mathcal{H}\text{om}(E, F)$ lies in the category $\mathfrak{B}_X^{\nu-\mu}$.*
- iii) *The category \mathfrak{B}_X^μ is abelian. For a point $x \in X(\mathbb{C}_p)$ consider the fibre functor ω_x on \mathfrak{B}_X^0 defined by $\omega_x(E) = E_x$. Then the pair $(\mathfrak{B}_X^0, \omega_x)$ is a neutral Tannakian category over \mathbb{C}_p .*
- iv) *Let $f : X' \rightarrow X$ be a finite morphism of smooth, projective, connected curves over $\overline{\mathbb{Q}}_p$ which has degree δ . Then a vector bundle E is in \mathfrak{B}_X^μ if and only if f^*E is in $\mathfrak{B}_{X'}^{\delta\mu}$.*
- v) *Any line bundle \mathcal{M} of degree d on $X_{\mathbb{C}_p}$ lies in \mathfrak{B}_X^d .*

Proof i) By [DW1], theorem 17 and theorem 13, all vector bundles in the category \mathfrak{B}_X^s are semistable of slope zero. Hence our claim follows from theorem 3.

- ii) By [DW1], theorem 17 and proposition 9, the category \mathfrak{B}_X^s is closed under tensor products and internal homs. Again the claim follows from theorem 3. A more direct proof is also possible using the fact that the class of vector bundles in definition 1 is closed under tensor products and internal homs.
- iii) By [DW2], corollary 12.3.4 the category \mathfrak{B}_X^{ps} is abelian. Using theorem 3, part iii) we find that for any slope μ the category \mathfrak{B}_X^μ is an additive full

subcategory of the category of all vector bundles on $X_{\mathbb{C}_p}$, which is closed under kernels and cokernels. Hence it is abelian.

By ii) the category \mathfrak{B}_X^0 is also stable under tensor product and internal homs, hence it is a neutral Tannaka category with respect to ω_x . Note that ω_x is faithful on the category of semistable vector bundles of degree zero on $X_{\mathbb{C}_p}$ and in particular on \mathfrak{B}_X^0 . Alternatively the faithfulness of ω_x on \mathfrak{B}_X^0 follows as in [DW1] Proposition 30 from the existence of the parallel transport for the bundles in \mathfrak{B}_X^0 c.f. Theorem 10 below.

iv) If $f : X' \rightarrow X$ is a finite morphism of degree δ as in the claim, then pullback of vector bundles induces a functor $\mathfrak{B}_{X'}^s \rightarrow \mathfrak{B}_X^s$ by [DW1], theorem 17 and proposition 9. By theorem 3, this implies our claim. One can also argue with the functoriality of the class of vector bundles in definition 1.

v) This follows by taking $\alpha = \text{id}$, $E = \mathcal{M}$ and $L = \mathcal{M}^*$ in theorem 3, ii). \square

Prompted by a question of Faltings in [Fa] and encouraged by Corollary 5 we make the following conjecture:

Conjecture 6 *Let X be a smooth projective curve over $\overline{\mathbb{Q}_p}$. Every semistable vector bundle of slope μ on $X_{\mathbb{C}_p}$ lies in \mathfrak{B}_X^μ .*

3 Ramified coverings of curves

For every variety Z over $\overline{\mathbb{Q}_p}$ we denote by $\Pi_1(Z)$ the following topological groupoid. Its set of objects is $Z(\mathbb{C}_p)$, and for objects $z, z' \in Z(\mathbb{C}_p)$ the set of morphisms $\text{Mor}_{\Pi_1(Z)}(z, z')$ is the set of isomorphisms of étale fibre functors $F_z \rightarrow F_{z'}$. Here F_z is the functor from the category of finite étale coverings of Z to the category of finite sets given by $F_z = \text{Mor}_Z(z, _)$. We call any such morphism an étale path (up to homotopy) from z to z' . As a profinite set $\text{Mor}_{\Pi_1(Z)}(z, z')$ carries a natural topology. If $z \in Z(\mathbb{C}_p)$ is a base point, we denote by $\pi_1(Z, z)$ the group of closed étale paths in z , i.e. the algebraic fundamental group of Z with base point z . Any morphism of varieties $f : Z \rightarrow Z'$ induces a natural functor $f_* : \Pi_1(Z) \rightarrow \Pi_1(Z')$.

For the purpose of descending the parallel transport we need the following construction:

Construction 7 Consider a finite étale Galois covering $\alpha : V \rightarrow U$ with Galois group G of varieties U and V over $\overline{\mathbb{Q}_p}$. Let $B : \Pi_1(V) \rightarrow \mathcal{C}$ be a functor to an abelian category \mathcal{C} . Assume that there is a system of isomorphisms $\varphi_\sigma : B \circ \sigma_* \xrightarrow{\sim} B$ for $\sigma \in G$ satisfying $\varphi_e = \text{id}$ and $\varphi_{\sigma\tau} = \varphi_\tau \circ (\varphi_\sigma \tau_*)$. Define a functor $A = A_B : \Pi_1(U) \rightarrow \mathcal{C}$ as follows: For $x \in \text{Ob } \Pi_1(U) = U(\mathbb{C}_p)$ set

$$A(x) = \{(f_y) \in \prod_{y \in \alpha^{-1}(x)} B(y) \mid \varphi_{\sigma,y}(f_{\sigma y}) = f_y \text{ for all } \sigma \in G \text{ and } y \in \alpha^{-1}(x)\}.$$

Let γ be an étale path in U from $x_1 \in U(\mathbb{C}_p)$ to $x_2 \in U(\mathbb{C}_p)$ and let y_1 be a point in $V(\mathbb{C}_p)$ with $\alpha(y_1) = x_1$. Then there is a unique path δ from y_1 to some point $y_2 \in V(\mathbb{C}_p)$ over x_2 such that $\alpha_*(\delta) = \gamma$. For $\sigma \in G$ the path $\sigma_*(\delta)$ is the unique lift of γ starting in σy_1 (and ending in σy_2). Applying the functor B gives an isomorphism $B(\sigma_*(\delta)) : B(\sigma y_1) \rightarrow B(\sigma y_2)$. The product of all $B(\sigma_*(\delta))$ induces an isomorphism

$$A(\gamma) : \prod_{y \in \alpha^{-1}(x_1)} B(y) = \prod_{\sigma \in G} B(\sigma y_1) \longrightarrow \prod_{\sigma \in G} B(\sigma y_2) = \prod_{y \in \alpha^{-1}(x_2)} B(y).$$

It maps $A(x_1)$ to $A(x_2)$ by the naturality of the φ_σ 's and it does not depend on the choice of the point y_1 over x_1 . Explicitely $A(\gamma)$ is given as follows. For an element $(f_{\sigma y_1})_{\sigma \in G}$ of $A(x_1)$ we have

$$(1) \quad A(\gamma)((f_{\sigma y_1})_{\sigma \in G}) = (B(\sigma_*(\delta))(f_{\sigma y_1}))_{\sigma \in G} \quad \text{in } A(x_2).$$

Here $B(\sigma_*(\delta))(f_{\sigma y_1})$ is in $B(\sigma y_2)$.

Having thus constructed a functor $A : \Pi_1(U) \rightarrow \mathcal{C}$ we define a canonical isomorphism of functors:

$$(2) \quad \Phi : A \circ \alpha_* \longrightarrow B.$$

Namely, for every $z \in V(\mathbb{C}_p)$ with $\alpha(z) = x$ the projection map

$$\prod_{y \in \alpha^{-1}(x)} B(y) \rightarrow B(z)$$

induces an isomorphism $\Phi(z) : A(x) \rightarrow B(x)$. This follows from the cocycle condition for the φ_σ 's above. For all $\sigma \in G$ the following diagram commutes:

$$(3) \quad \begin{array}{ccc} A \circ \alpha_* \circ \sigma_* & \xrightarrow{\Phi \circ \sigma_*} & B \circ \sigma_* \\ \downarrow \wr \text{can} & & \downarrow \varphi_\sigma \\ A \circ \alpha_* & \xrightarrow{\Phi} & B. \end{array}$$

If \mathcal{C} is a topological abelian category and B above a continuous functor, then the functor A is continuous as well.

Remark The explicit construction of the functor $A = A_B$ and our later calculations with it are somewhat clumsy. However since we have to compute group actions later and since non-trivial actions could hide in isomorphisms we felt more secure with this pedestrian setup.

As before we fix a smooth, projective and connected curve X over $\overline{\mathbb{Q}}_p$ and write $X_{\mathbb{C}_p}$ for the base change of X to \mathbb{C}_p . We consider a finite Galois covering

$$\alpha : Y \longrightarrow X$$

of X with Galois group G , by a smooth, projective and connected curve Y over $\overline{\mathbb{Q}}_p$. By $S \subset X(\overline{\mathbb{Q}}_p)$ we denote the ramification locus of α , and we put $T = \alpha^{-1}(S) \subset Y(\overline{\mathbb{Q}}_p)$. Moreover, let $U = X \setminus S$ and $V = Y \setminus T$. We denote by $j : V \hookrightarrow Y$ the corresponding open immersion.

Note that the étale Galois covering $\alpha|_V : V \rightarrow U$ gives rise to an exact sequence

$$(4) \quad 1 \rightarrow \pi_1(V, y_0) \rightarrow \pi_1(U, x_0) \rightarrow G \rightarrow 1,$$

where $y_0 \in V(\mathbb{C}_p)$ and $x_0 = \alpha(y_0)$ are base points.

Throughout this section we fix a vector bundle F on $Y_{\mathbb{C}_p}$ carrying a G -action, i.e. we have isomorphisms

$$a_\sigma : \sigma^* F \xrightarrow{\sim} F \text{ for all } \sigma \in G$$

satisfying $a_{\sigma\tau} = a_\tau \circ \tau^*(a_\sigma)$ and $a_e = \text{id}$. Assume that F lies in the category \mathfrak{B}_Y^{ps} .

In [DW1], Theorem 36 and remark, we defined a continuous functor ρ_F from $\Pi_1(Y)$ to the category $\mathbf{Vec}_{\mathbb{C}_p}$ of finite dimensional \mathbb{C}_p -vector spaces. For every $z \in Y(\mathbb{C}_p)$ we have $\rho_F(z) = F_z$, where F_z is the fibre of F in z . For a morphism γ from y_1 to y_2 , i.e. an étale path from y_1 to y_2 , the isomorphism $\rho_F(\gamma) : F_{y_1} \rightarrow F_{y_2}$ is called the parallel transport along γ . The association $F \mapsto \rho_F$ is functorial. In particular, if σ is an element of G , there is a commutative diagram

$$(5) \quad \begin{array}{ccc} F_{\sigma(y_1)} & \xrightarrow{a_{\sigma,y_1}} & F_{y_1} \\ \rho_{\sigma^* F}(\gamma) \downarrow & & \downarrow \rho_F(\gamma) \\ F_{\sigma(y_2)} & \xrightarrow{a_{\sigma,y_2}} & F_{y_2}. \end{array}$$

Since our construction is compatible with pullbacks by [DW1], theorem 36, we have $\rho_F \circ \sigma_* = \rho_{\sigma^* F}$, where $\sigma_* : \Pi_1(Y) \rightarrow \Pi_1(Y)$ is induced by σ . The system of isomorphisms of continuous functors from $\Pi_1(V)$ to $\mathbf{Vec}_{\mathbb{C}_p}$

$$\varphi_\sigma : (\rho_F \circ j_*) \circ \sigma_* = (\rho_F \circ \sigma_*) \circ j_* = \rho_{\sigma^* F} \circ j_* \xrightarrow{a_\sigma \circ j_*} \rho_F \circ j_*$$

satisfies the cocycle condition of construction 7. Hence we get a continuous functor

$$(6) \quad \rho = A_{\rho_F \circ j_*} : \Pi_1(U) \longrightarrow \mathbf{Vec}_{\mathbb{C}_p}$$

and a canonical isomorphism $\Phi : \rho \circ \alpha_* \xrightarrow{\sim} \rho_F \circ j_*$ satisfying $\varphi_\sigma \circ (\Phi \circ \sigma_*) = \Phi$ for every $\sigma \in G$.

In particular, for every base point $x_0 \in U(\mathbb{C}_p)$ we get a continuous representation

$$(7) \quad \rho : \pi_1(U, x_0) \rightarrow \mathrm{GL}(\rho(x_0)).$$

Now we want to show that ρ factors over a certain quotient of $\pi_1(U, x_0)$ by studying the monodromy along étale paths around points in the ramification divisor S . For every closed point y in Y we denote by G_y the subgroup of G consisting of all elements fixing y . For $\sigma \in G_y$ the map $a_\sigma : \sigma^* F \xrightarrow{\sim} F$ has fibre $a_{\sigma, y} : F_y = (\sigma^* F)_y \rightarrow F_y$, an automorphism of F_y . The map $G_y \rightarrow \mathrm{GL}(F_y)$ sending σ to $a_{\sigma, y}$ is an antihomomorphism.

Lemma 8 *Let t be a point in T (viewed in $T(\mathbb{C}_p)$) and assume that for all $\sigma \in G_t$ the element $a_{\sigma, t} \in \mathrm{GL}(F_t)$ is central, i.e. there exists a character*

$$\chi_t : G_t \rightarrow \mathbb{C}_p^\times$$

satisfying $a_{\sigma, t} = \chi_t(\sigma) \mathrm{id}_{F_t}$. We fix a base point x_0 in $U(\mathbb{C}_p)$ and a preimage y_0 of x_0 in $V(\mathbb{C}_p)$. Let γ_0 be an element in $\pi_1(U, x_0)$ mapping via (4) to an element $\tau \in G_t$, and let γ be an étale path in V with starting point y_0 that lifts γ_0 . Its endpoint is therefore $\tau(y_0)$. If there exists an étale path δ in Y from y_0 to t satisfying

$$(*) \quad \rho_F(\tau_*(\delta) j_*(\gamma) \delta^{-1}) = \mathrm{id}_{F_t},$$

then we have

$$\rho(\gamma_0) = \chi_t(\tau) \mathrm{id}_{\rho(x_0)}.$$

If $()$ holds for one étale path δ from y_0 to t , it holds for all such paths.*

Remark We will verify condition $(*)$ in the proof of theorem 9 below using Grothendieck's comparison theorem between algebraic and topological fundamental groups. We do not know an algebraic argument showing that $(*)$ is satisfied.

Proof Since ρ_F is a functor, we find

$$\begin{aligned} \text{id}_{F_t} &= \rho_F(\tau_*(\delta)j_*(\gamma)\delta^{-1}) \\ &= \rho_F(\tau_*(\delta)) \circ \rho_F(j_*(\gamma)) \circ \rho_F(\delta^{-1}), \end{aligned}$$

hence $\rho_F(j_*(\gamma)) = \rho_F(\tau_*(\delta))^{-1} \circ \rho_F(\delta)$. Now according to (5) we have

$$\rho_F(\tau_*(\delta)) = \rho_{\tau^*F}(\delta) = a_{\tau,t}^{-1} \circ \rho_F(\delta) \circ a_{\tau,y_0}.$$

By assumption, we have $a_{\tau,t} = \chi_t(\tau)\text{id}_{F_t}$, hence

$$\rho_F(j_*(\gamma)) = \rho_F(\tau_*(\delta))^{-1} \circ \rho_F(\delta) = a_{\tau,y_0}^{-1} \circ \rho_F(\delta)^{-1} \circ a_{\tau,t} \circ \rho_F(\delta) = \chi_t(\tau)a_{\tau,y_0}^{-1}.$$

Hence for every element (f_y) of

$$\rho(x_0) = \{(f_y) \in \prod_{y \in \alpha^{-1}(x_0)} F_y \mid a_{\sigma,y}(f_{\sigma y}) = f_y \text{ for all } \sigma \in G\}$$

we find

$$\rho_F(j_*(\gamma))(f_{y_0}) = \chi_t(\tau)a_{\tau,y_0}^{-1}(f_{y_0}) = \chi_t(\tau)f_{\tau y_0}.$$

This implies that $\rho(\gamma_0) = \chi_t(\tau)\text{id}_{\rho(x)}$. Namely, according to (1) the component of $\rho(\gamma_0)(f_{\sigma y_0})_{\sigma \in G}$ in $F_{\sigma \tau y_0}$ is given by

$$\begin{aligned} \rho_F(\sigma_*(j_*(\gamma)))(f_{\sigma y_0}) &= a_{\sigma,\tau y_0}^{-1}(\rho_F(j_*(\gamma))(f_{y_0})) \\ &= a_{\sigma,\tau y_0}^{-1}(\chi_t(\tau)f_{\tau y_0}) \\ &= \chi_t(\tau)a_{\sigma,\tau y_0}^{-1}(f_{\tau y_0}) \\ &= \chi_t(\tau)f_{\sigma \tau y_0}. \end{aligned}$$

This shows that as claimed we have

$$\rho(\gamma_0)((f_y)) = \chi_t(\tau)(f_y)$$

where $(f_y) = (f_y)_{y \in \alpha^{-1}(x_0)}$.

Finally, we check that the condition $(*)$ is independent of the choice of δ . If δ' is a second étale path in Y from y_0 to t , then $\eta = \delta'\delta^{-1}$ is an element of $\pi_1(Y, t)$. Since $a_{\tau, t}$ is central, we deduce from (5) that we have

$$\rho_F(\tau_*(\eta)) = \rho_{\tau^*F}(\eta) = a_{\tau, t}^{-1} \circ \rho_F(\eta) \circ a_{\tau, t} = \rho_F(\eta).$$

Hence, using condition $(*)$ for δ , we find

$$\begin{aligned} \rho_F(\tau_*(\delta')j_*(\gamma)(\delta')^{-1}) &= \rho_F(\tau_*(\eta\delta)j_*(\gamma)\delta^{-1}\eta^{-1}) \\ &= \rho_F(\tau_*(\eta))\rho_F(\eta)^{-1} \\ &= \text{id}_{F_t}, \end{aligned}$$

hence $(*)$ also holds for δ' . \square

Now we use the transcendental theory of the fundamental group. Fix an embedding $\mathbb{C}_p \subset \mathbb{C}$ of abstract fields and set $X_{\mathbb{C}} = X \otimes_{\overline{\mathbb{Q}_p}} \mathbb{C}$ etc. By $X^{\text{an}} = (X_{\mathbb{C}})^{\text{an}}$ etc. we denote the associated Riemann surfaces. Any point $x_0 \in U(\mathbb{C}_p)$ then defines points x_0 in $U_{\mathbb{C}}(\mathbb{C})$ and U^{an} . By [SGA1] exposé X, Corollaire 1.8 and exposé XII, Corollaire 5.2 we have canonical isomorphisms

$$\hat{\pi}_1(U^{\text{an}}, x_0) \xrightarrow{\sim} \pi_1(U_{\mathbb{C}}, x_0) \xrightarrow{\sim} \pi_1(U, x_0).$$

Here $\hat{\pi}_1(U^{\text{an}}, x_0)$ denotes the profinite completion of the topological fundamental group $\pi_1(U^{\text{an}}, x_0)$. Let g be the genus of X . For every $s \in S \cong S_{\mathbb{C}}$ we choose a closed loop γ_s with basepoint x_0 in U^{an} such that γ_s is contained in a contractible open subset of $U^{\text{an}} \cup \{s\}$ and winds once around s , counterclockwise with respect to the orientation. Moreover, for all $i = 1, \dots, g$ let γ_i and γ_{g+i} be closed loops in U^{an} with basepoint x_0 such that $\pi_1(X^{\text{an}}, x_0)$ is the group generated by $\gamma_1, \dots, \gamma_{2g}$ subject to the single relation

$$\prod_{i=1}^g [\gamma_i, \gamma_{g+i}] = 1.$$

Then $\pi_1(U^{\text{an}}, x_0)$ is the group generated by $\gamma_1, \dots, \gamma_{2g}$ and all γ_s for $s \in S$ subject to the relation (where S carries a suitable ordering)

$$\prod_{s \in S} \gamma_s \prod_{i=1}^g [\gamma_i, \gamma_{g+i}] = 1.$$

By abuse of notation we denote by γ_i and γ_s also the corresponding elements in the algebraic fundamental group under the map

$$\pi_1(U^{\text{an}}, x_0) \longrightarrow \hat{\pi}_1(U^{\text{an}}, x_0) \cong \pi_1(U, x_0)$$

and similarly for $\pi_1(X, x_0)$. Then $\pi_1(X, x_0)$ is the profinite group generated by all γ_i subject to the relation $\prod_{i=1}^g [\gamma_i, \gamma_{i+g}] = 1$. Set

$$N = \text{Ker}(\pi_1(U, x_0) \longrightarrow \pi_1(X, x_0)) .$$

Then N is the closed normal subgroup of $\pi_1(U, x_0)$ generated by the elements γ_s for $s \in S$. Note that γ_s generates an inertia group I_s at s . Let $[N, \pi_1(U, x_0)]$ be the closed normal subgroup of $\pi_1(U, x_0)$ generated by all commutators $[n, \gamma]$ for $\gamma \in \pi_1(U, x_0)$ and $n \in N$ or equivalently for $n = \gamma_s$ for $s \in S$. Setting

$$\tilde{N} = N/[N, \pi_1(U, x_0)] \quad \text{and} \quad \tilde{\Gamma} = \pi_1(U, x_0)/[N, \pi_1(U, x_0)] ,$$

we get a canonical central extension

$$1 \rightarrow \tilde{N} \rightarrow \tilde{\Gamma} \rightarrow \pi_1(X, x_0) \rightarrow 1 .$$

All inertia groups at $s \in S$ in N have the same image \tilde{I}_s in \tilde{N} because \tilde{N} is central in $\tilde{\Gamma}$. Moreover, by Kummer theory each \tilde{I}_s is canonically isomorphic to $\hat{\mathbb{Z}}(1) = \varprojlim_n \mu_n$ where $\mu_n = \mu_n(\mathbb{C}_p)$. The image $\tilde{\gamma}_s$ of γ_s in \tilde{N} is a generator of \tilde{I}_s . For $g \geq 1$ we have

$$\tilde{N} = \prod_{s \in S} \tilde{I}_s = \hat{\mathbb{Z}}(1)^{|S|} .$$

For $g = 0$ the group \tilde{N} is the quotient of $\prod_{s \in S} \tilde{I}_s$ by the single relation $\prod_{s \in S} \tilde{\gamma}_s = 1$ and hence it is isomorphic to $\hat{\mathbb{Z}}(1)^{|S|-1}$.

Now fix integers $d_s \geq 1$ for $s \in S$ and let Γ be the quotient of $\tilde{\Gamma}$ by the relations $\gamma^{d_s} = 1$ for all $\gamma \in \tilde{I}_s$, ($\gamma = \gamma_s$ suffice) and $s \in S$.

The image \bar{I}_s of \tilde{I}_s in Γ is canonically isomorphic to μ_{d_s} and generated by $\bar{\gamma}_s$, the image of $\tilde{\gamma}_s$ in $\bar{\Gamma}$. There is an exact sequence

$$(8) \quad 1 \rightarrow \bar{N} \rightarrow \Gamma \rightarrow \pi_1(X, x_0) \rightarrow 1$$

where

$$\bar{N} = \prod_{s \in S} \bar{I}_s = \prod_{s \in S} \mu_{d_s} \quad \text{if } g \geq 1 .$$

For $g = 0$ the group Γ is the quotient of $\prod_{s \in S} \bar{I}_s$ by the single relation $\prod_{s \in S} \bar{\gamma}_s = 1$.

We now have the following result:

Theorem 9 Consider as above a Galois covering $\alpha : Y \rightarrow X$ with group G , ramification loci $S \subset X$ and $T = \alpha^{-1}(S) \subset Y$ with complements $U = X \setminus S$ and $V = Y \setminus T$. For a vector bundle F in \mathfrak{B}_Y^{ps} with G -action given by $a_\sigma : \sigma^*F \rightarrow F$ for every $\sigma \in G$ and for a point $x_0 \in U(\mathbb{C}_p)$ consider the representation (7) obtained by descending ρ_F :

$$\rho : \pi_1(U, x_0) \longrightarrow \mathrm{GL}(\rho(x_0)).$$

Assume that for every point $t \in T$ and every $\sigma \in G_t$ the element $a_{\sigma, t} \in \mathrm{GL}(F_t)$ is central. Let $\chi_t : G_t \rightarrow \mu \subset \mathbb{C}_p^*$ be the corresponding character satisfying $a_{\sigma, t} = \chi_t(\sigma)\mathrm{id}_{F_t}$. Given $s \in S$, let $t \in T$ be a point with $\alpha(t) = s$. Then there exists a generator τ of G_t such that $\rho(\gamma_s)$ is the automorphism

$$\rho(\gamma_s) = \chi_t(\tau)\mathrm{id} : \rho(x_0) \longrightarrow \rho(x_0).$$

The order o_s of the character χ_t depends only on s and divides the ramification index e_s of the point s . Fix integer multiples $d_s \geq 1$ of o_s for $s \in S$ and let Γ be the quotient of $\pi_1(U, x_0)$ constructed above using the numbers d_s for $s \in S$. Then the representation ρ factors over Γ and induces a continuous representation

$$\rho : \Gamma \longrightarrow \mathrm{GL}(\rho(x_0)).$$

Proof We look at the analytic covering $\alpha^{an} : Y^{an} \rightarrow X^{an}$ induced by α . It is unramified over U^{an} . For any $s \in S$ choose a contractible open non-compact neighbourhood V_s of γ_s in $U^{an} \cup \{s\}$ which contains s . Shrinking V_s if necessary we may assume that V_s is not biholomorphic to \mathbb{C} . By Riemann's mapping theorem there is then a biholomorphic map from V_s to the open unit disc E , mapping s to 0. Let W be the connected component of $(\alpha^{an})^{-1}V_s$ containing t . By [Fo], Satz 5.11, there exists a biholomorphic map $\varphi : W \rightarrow E$ such that the diagram

$$\begin{array}{ccc} W & \xrightarrow{\varphi} & E \\ \alpha^{an} \downarrow & & \downarrow z \mapsto z^{e_s} \\ V_s & \xrightarrow{\sim} & E \end{array}$$

commutes. In particular we have $\varphi(t) = 0$. Choose a point $y_0 \in W$ mapping to x_0 and lift γ_s to a path γ in $W \setminus \{t\}$ with starting point y_0 . Its endpoint is a point in W of the form τy_0 for some $\tau \in G$ of order e_s . Note here that γ_s winds around s once. Hence τ maps the connected component W to itself,

which implies $\tau t = t$, i.e. τ is contained in G_t . Since $|G_t| = e_s$ it follows that τ generates G_t .

Choose a path δ in W from y_0 to t . Then $\tau(\delta)\gamma\delta^{-1}$ is a closed path in W . Since W is biholomorphic to the unit disc E , this path is null-homotopic in W , and hence also in Y^{an} . The topological path δ induces an étale path in Y connecting y_0 and t . By abuse of notation, we denote it also by δ . It follows that the composition $\tau_*(\delta)j_*(\gamma)\delta^{-1}$ is trivial in the algebraic fundamental group $\pi_1(Y, t)$. Hence we can apply lemma 8 to conclude that

$$\rho(\gamma_s) = \chi_t(\tau)\text{id}_{\rho(x_0)}.$$

In particular, $\rho(\gamma_s)$ is central in $\text{GL}(\rho(x_0))$ for every $s \in S$. Therefore $\rho : \pi_1(U, x_0) \rightarrow \text{GL}(\rho(x_0))$ factors over $\tilde{\Gamma}$. Given two points t, t' over s there is some $\pi \in G$ with $\pi t = t'$. The cocycle relation for the a_σ 's implies that we have a commutative diagram

$$\begin{array}{ccc} G_t & \xrightarrow{\sim} & G_{t'} \\ \chi_t \searrow & & \swarrow \chi_{t'} \\ & \mu & \end{array}$$

where the horizontal map is given by conjugation with π . Hence the orders of χ_t and $\chi_{t'}$ are the same and divide $e_s = |G_t|$. We have $\rho(\gamma_s^{d_s}) = \rho(\gamma_s)^{d_s} = \chi_t(\tau)^{d_s} = 1$ since $o_s \mid d_s$, so that indeed ρ induces a representation

$$\rho : \tilde{\Gamma} \longrightarrow \text{GL}(\rho(x_0)).$$

Note here that $\tilde{\gamma}_s$ generates \tilde{I}_s for every $s \in S$. \square

Theorem 9 will be used in the next section to define representations for certain bundles of non-zero slope. For now we use it to define a parallel transport for bundles in \mathfrak{B}_X^0 , the category of vector bundles of slope zero on X with potentially strongly semistable reduction in the sense of definition 2.

Theorem 10 *Let E be a vector bundle on $X_{\mathbb{C}_p}$ contained in the category \mathfrak{B}_X^0 . The construction below gives a continuous functor*

$$\rho_E : \Pi_1(X) \longrightarrow \text{Vec}_{\mathbb{C}_p}$$

from the étale fundamental groupoid $\Pi_1(X)$ to the category of finite-dimensional \mathbb{C}_p -vector spaces, mapping every $x \in X(\mathbb{C}_p)$ to the fibre E_x . In other words,

there is a functorial parallel transport on E along étale paths on X . For bundles in \mathfrak{B}_X^0 parallel transport is compatible with $\otimes, \underline{\text{Hom}}, \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ -conjugation and pullbacks via morphisms of smooth projective curves over $\overline{\mathbb{Q}}_p$. In particular, for all $x \in X(\mathbb{C}_p)$, we obtain a representation $\rho_{E,x} : \pi_1(X, x) \rightarrow \text{GL}(E_x)$ of the étale fundamental group satisfying the corresponding compatibilities. For bundles in \mathfrak{B}_X^{ps} the parallel transport is the same as the one defined in [DW1].

Proof Since E lies in \mathfrak{B}_X^0 , by theorem 3 there exists a finite morphism $\alpha : Y \rightarrow X$ of smooth, projective, connected curves over $\overline{\mathbb{Q}}_p$, such that $F = \alpha_{\mathbb{C}_p}^* E$ lies in \mathfrak{B}_Y^{ps} . We may assume that α is Galois with group G . Then we have canonical identifications $a_\sigma : \sigma^* F = F$ satisfying the cocycle conditions and hence we get the descended functor (6):

$$\rho = A_{\rho_F \circ j_*} : \Pi_1(U) \longrightarrow \mathbf{Vec}_{\mathbb{C}_p} .$$

For $x \in \text{Ob } \Pi_1(U) = U(\mathbb{C}_p)$ we have

$$\begin{aligned} \rho(x) &= \{(f_y) \in \prod_{\alpha(y)=x} F_y \mid a_{\sigma,y}(f_{\sigma y}) = f_y \text{ for all } \sigma \in G\} \\ &= \{(f_y) \in \prod_{\alpha(y)=x} E_x \mid f_{\sigma y} = f_y \text{ for all } \sigma \in G\} . \end{aligned}$$

Hence we may identify $\rho(x)$ and E_x . Clearly, for $\sigma \in G_t$ the action of $a_{\sigma,t}$ on F_t is given by the trivial character. For any $x_0 \in U(\mathbb{C}_p)$ applying theorem 9, we find that $\rho(\gamma_s) = 1$. Therefore the representation $\pi_1(U, x_0) \longrightarrow \text{GL}(E_{x_0})$ induced by ρ factors through $\pi_1(X, x_0)$.

Let $\lambda : U \hookrightarrow X$ be the open immersion. Recall that $j : V \hookrightarrow Y$ denotes the corresponding map on the covering. Now let γ be any étale path on X from $x_1 \in X(\mathbb{C}_p)$ to $x_2 \in X(\mathbb{C}_p)$. We claim that γ can be lifted to an étale path in Y . Choose preimages y_1 and y_2 of x_1 and x_2 in $Y(\mathbb{C}_p)$. By [SGA1], exposé V, no. 7, there exists an étale path ϵ in Y connecting y_2 and y_1 . Then $\alpha_*(\epsilon)\gamma$ is a closed étale path in X based in x_1 . If x_1 lies in $U(\mathbb{C}_p)$, then $\alpha_*(\epsilon)\gamma = \lambda_*(\delta)$ for some closed étale path δ on U , since $\pi_1(U, x_1) \rightarrow \pi_1(X, x_1)$ is surjective. Since $\alpha : V \rightarrow U$ is étale, there exists an étale path δ' on V lifting δ with endpoint y_1 . Then

$$\alpha_*(\epsilon^{-1} j_*(\delta')) = \alpha_*(\epsilon)^{-1} \lambda_* \alpha_*(\delta') = \alpha_*(\epsilon)^{-1} \lambda_*(\delta) = \alpha_*(\epsilon)^{-1} \alpha_*(\epsilon)\gamma = \gamma .$$

Hence any path γ starting in a point of $U(\mathbb{C}_p)$ can be lifted to an étale path in Y . If x_1 does not lie in $U(\mathbb{C}_p)$, we choose a point $x_0 \in U(\mathbb{C}_p)$, a preimage y_0 of x_0 in $V(\mathbb{C}_p)$ and an étale path δ_1 from y_0 to y_1 in Y . Then $\gamma\alpha_*(\delta_1)$ is an étale path in X starting in x_0 . As we have just shown, it can be lifted to an étale path in Y . This implies that γ can be lifted to an étale path in Y , too.

We claim that there is a representation

$$\rho_E : \Pi_1(X) \longrightarrow \mathbf{Vec}_{\mathbb{C}_p}$$

such that ρ above factors as

$$\rho : \Pi_1(U) \rightarrow \Pi_1(X) \xrightarrow{\rho_E} \mathbf{Vec}_{\mathbb{C}_p} .$$

We define ρ_E by setting $\rho_E(x) = E_x$ for all $x \in X(\mathbb{C}_p)$ and by

$$\rho_E(\gamma) = \rho_F(\delta),$$

if δ is an arbitrary étale path in Y satisfying $\alpha_*(\delta) = \gamma$. Here we identify F_y with E_x for any point y above x .

We have already shown that such a lift δ always exists. It remains to check that the definition is independent of the choice of δ . Let γ be an étale path in X starting in $x_1 \in U(\mathbb{C}_p)$, and assume that δ_1 and δ_2 are étale paths in Y lifting γ . We denote by $y_1 \in V(\mathbb{C}_p)$ the starting point of δ_1 . Replacing δ_2 by $\sigma_*\delta_2$ for a suitable $\sigma \in G$, we may assume that δ_1 and δ_2 have the same endpoint. Hence $\delta_2^{-1}\delta_1$ is an étale path in Y connecting two points in V . Since for all $y_0 \in V(\mathbb{C}_p)$ the map $\pi_1(V, y_0) \rightarrow \pi_1(Y, y_0)$ is surjective, there exists an étale path ϵ in V such that $j_*\epsilon = \delta_2^{-1}\delta_1$. Hence

$$\rho_F(\delta_2)^{-1}\rho_F(\delta_1) = \rho_F(\delta_2^{-1}\delta_1) = \rho_F(j_*(\epsilon)) = \rho(\alpha_*(\epsilon))$$

by definition of ρ . Now $\alpha_*(\epsilon)$ is a closed étale path in U satisfying $\lambda_*\alpha_*(\epsilon) = \alpha_*j_*(\epsilon) = \alpha_*(\delta_2^{-1}\delta_1) = 1$. Therefore it lies in the kernel of $\pi_1(U, x_1) \rightarrow \pi_1(X, x_1)$, which implies $\rho(\alpha_*(\epsilon)) = 1$, as we have shown at the beginning of the proof.

It remains to treat the case that the starting point x_1 of γ does not belong to U . Assume that δ_1 and δ_2 are étale paths in Y lifting γ . Let $y_1 \in Y(\mathbb{C}_p)$ be the starting point of δ_1 . Replacing δ_2 by $\sigma_*(\delta_2)$ for a suitable $\sigma \in G$, we can assume that δ_2 also starts in y_1 . Choose a point $x_0 \in U(\mathbb{C}_p)$, a preimage $y_0 \in V(\mathbb{C}_p)$

of x_0 and an étale path ϵ in Y from y_0 to y_1 . Then $\delta_1\epsilon$ and $\delta_2\epsilon$ are paths in Y lifting the étale path $\gamma\alpha_*\epsilon$. Since $\gamma\alpha_*\epsilon$ starts in $x_0 \in U(\mathbb{C}_p)$, we have already shown that $\rho_F(\delta_1\epsilon) = \rho_F(\delta_2\epsilon)$, which implies $\rho_F(\delta_1) = \rho_F(\delta_2)$.

Using the corresponding results from [DW1] it follows that ρ_E is a functor which is functorial in E and satisfies the stated compatibilities. \square

Remark 11 *In the construction of the functor $\rho = A_{\rho_F \circ j_*}$ in (6) we had assumed that F was a vector bundle in \mathfrak{B}_Y^{ps} in order to have the parallel transport ρ_F . Using theorem 10 one can consider the functor $\rho = A_{\rho_F \circ j_*}$ also for the bundles in the a priori bigger category \mathfrak{B}_Y^0 and lemma 8 and theorem 9 continue to hold with identical proofs, if F is in \mathfrak{B}_Y^0 .*

4 Vector bundles of non-vanishing slope

As usual, let X be a smooth, projective and connected curve over $\overline{\mathbb{Q}}_p$. We have seen in the last section that bundles in \mathfrak{B}_X^0 admit functorial isomorphisms of parallel transport. In particular, they give rise to p -adic representations of the fundamental group $\pi_1(X, x_0)$. Now we will look at bundles in \mathfrak{B}_X^μ for arbitrary slopes $\mu = d/r$. In general we cannot expect p -adic representations of the fundamental group. However, similarly to the classical case we will define for any bundle E in \mathfrak{B}_X^μ a representation ρ_{E,x_0} of a certain central extension Γ_r of $\pi_1(X, x_0)$, c.f. [NS], proposition 6.2 and [AB], section 6. We hope to study these representations elsewhere.

In [NS] the construction uses an infinite extension of the base Riemann surface which is ramified in exactly one point. In our algebraic situation we might naively try to use \mathbb{Z}/r -coverings which are ramified in one point of X only. However such coverings do not exist for $r \geq 2$. Coverings with group \mathbb{Z}/r which are ramified in exactly two points do exist though and we can use them for our purposes.

Construction 12 *Fix an integer $r \geq 1$ and two different points s, s_0 in $X(\overline{\mathbb{Q}}_p)$. Below we construct finite étale coverings $\alpha : Y \longrightarrow X$ by smooth, projective, connected $\overline{\mathbb{Q}}_p$ -curves Y such that*

- α is Galois with Galois group $G = \mu_r$,
- α is unramified outside $\{s, s_0\}$ and
- α is totally ramified over s and s_0 .

Proof Put $U = X \setminus \{s, s_0\}$ and consider the following relative exact sequence of étale cohomology groups together with the trace isomorphism:

$$H^1(X, \mu_r) \longrightarrow H^1(U, \mu_r) \xrightarrow{\delta} H^2_{\{s, s_0\}}(X, \mu_r) \longrightarrow H^2(X, \mu_r) \xrightarrow{\text{tr}} \mathbb{Z}/r.$$

Besides we have:

$$H^2_{\{s, s_0\}}(X, \mu_r) = H^2_s(X, \mu_r) \oplus H^2_{s_0}(X, \mu_r) \simeq \mathbb{Z}/r \oplus \mathbb{Z}/r,$$

and these identifications make the following diagram commutative

$$\begin{array}{ccccc} H^1(U, \mu_r) & \xrightarrow{\delta} & H^2_{\{s, s_0\}}(X, \mu_r) & \longrightarrow & H^2(X, \mu_r) \\ & & \downarrow \simeq & & \downarrow \simeq \\ & & \mathbb{Z}/r \oplus \mathbb{Z}/r & \xrightarrow{\Sigma} & \mathbb{Z}/r \end{array}$$

Fix a point $x_0 \in U(\mathbb{C}_p)$ and choose a class $c \in H^1(U, \mu_r) = \text{Hom}(\pi_1(U, x_0), \mu_r)$ such that $\delta(c)$ corresponds to $(1, -1) \in \mathbb{Z}/r\mathbb{Z} \oplus \mathbb{Z}/r\mathbb{Z}$. The image of the homomorphism

$$c : \pi_1(U, x_0) \longrightarrow \mu_r$$

is a subgroup μ_t of μ_r for some t dividing r . Then tc is trivial, which implies $(t, -t) = t\delta(c) = 0$ in $\mathbb{Z}/r\mathbb{Z} \times \mathbb{Z}/r\mathbb{Z}$. Hence $t = r$, i.e. c is surjective. Its kernel defines a connected Galois étale covering $\alpha : V \rightarrow U$ with Galois group μ_r . Let Y be the normalization of X in the function field of V . Then α extends uniquely to a finite morphism $\alpha : Y \rightarrow X$, where Y is a connected, smooth and projective curve over $\overline{\mathbb{Q}}_p$.

Under pullback by α , the class c vanishes in $H^1(V, \mu_r)$. Hence $\delta(\alpha^*(c))$ is zero in

$$H^2_{\alpha^{-1}\{s, s_0\}}(Y, \mu_r) = H^2_{\alpha^{-1}(s)}(Y, \mu_r) \oplus H^2_{\alpha^{-1}(s_0)}(Y, \mu_r).$$

Denoting by e the common ramification index of the points in $\alpha^{-1}(s)$ we have a commutative diagram with $\Delta_e(a) = (ea, \dots, ea)$:

$$\begin{array}{ccccc} H^2_{\alpha^{-1}(s)}(Y, \mu_r) & \xlongequal{\quad} & \bigoplus_{t \in \alpha^{-1}(s)} H^2_t(Y, \mu_r) & \xlongequal{\quad} & \bigoplus_{t \in \alpha^{-1}(s)} \mathbb{Z}/r \\ \uparrow \alpha^* & & & & \uparrow \Delta_e \\ H^2_s(X, \mu_r) & \xrightarrow{\sim} & & & \mathbb{Z}/r \end{array}$$

Since $\delta(\alpha^*(c))$ vanishes, we have $\Delta_e(1) = 0$ and hence $e = r$. Thus α is totally ramified over s and by the same argument also over s_0 . \square

Now we construct certain central extensions of $\pi_1(X, x_0)$. With notation as in the preceding section set $d_s = r$ and $d_{s_0} = 1$ and write Γ_r for Γ . For $g = 0$ we have $\Gamma_r = \bar{N} = 1$. For $g \geq 1$ there is the central extension (8) where we have identified $\bar{N} = \bar{I}_s$ with μ_r

$$(9) \quad 1 \longrightarrow \mu_r \longrightarrow \Gamma_r \longrightarrow \pi_1(X, x_0) \longrightarrow 1 .$$

One can show that this extension corresponds to $1 \in \mathbb{Z}/r$ under the canonical isomorphisms

$$H^2(\pi_1(X, x_0), \mu_r) = H^2(X, \mu_r) = \mathbb{Z}/r .$$

We elaborate on this a bit:

Digression (without proofs) More generally let $S \subset X$ be a finite set of closed points and set $U = X \setminus S$. Assuming $g \geq 1$ let d_2 be the transgressions in the Hochschild–Serre spectral sequence for the extension

$$(10) \quad 1 \longrightarrow N \longrightarrow \pi_1(U, x_0) \longrightarrow \pi_1(X, x_0) \longrightarrow 1$$

and in the Leray spectral sequence for the inclusion $\lambda : U \hookrightarrow X$. Then we have a canonical commutative diagram

$$\begin{array}{ccc} H^0(\pi_1(X, x_0), H^1(N, \mu_r)) & \xrightarrow{d_2} & H^2(\pi_1(X, x_0), \mu_r) \\ \parallel & & \parallel \\ H^0(X, R^1\lambda_*\mu_r) & \xrightarrow{d_2} & H^2(X, \mu_r) \\ \downarrow \wr & & \parallel \\ H_S^2(X, \mu_r) & \longrightarrow & H^2(X, \mu_r) . \end{array}$$

Using the identifications

$$H^0(\pi_1(X, x_0), H^1(N, \mu_r)) = \text{Hom}_{\pi_1(X, x_0)}(N^{\text{ab}}, \mu_r) = \text{Hom}(\tilde{N}, \mu_r)$$

where \tilde{N} was defined before theorem 9 we get a commutative diagram

$$\begin{array}{ccc} \text{Hom}(\tilde{N}, \mu_r) & \xrightarrow{d_2} & H^2(\pi_1(X, x_0), \mu_r) \\ \omega \downarrow & & \parallel \\ \bigoplus_{s \in S} \mathbb{Z}/r & \xrightarrow{\Sigma} & \mathbb{Z}/r . \end{array}$$

Here for a homomorphism $\varphi : \tilde{N} \rightarrow \mu_r$ the element $d_2(\varphi)$ is the class of the push-out of (10) via φ . The map ω is given as follows. Identifying the inertia groups at the points $s \in S$ with μ_r we have $\tilde{N} = \prod_{s \in S} \mu_r$. Then ω is the composition:

$$\mathrm{Hom}(\tilde{N}, \mu_r) = \bigoplus_{s \in S} \mathrm{Hom}(\mu_r, \mu_r) = \bigoplus_{s \in S} \mathbb{Z}/r .$$

The extension (9) for example is the push-out of (10) via the map $\varphi \in \mathrm{Hom}(\tilde{N}, \mu_r)$ corresponding to $\omega(\varphi) = (1, 0)$. Hence it corresponds to $\Sigma(1, 0) = 1 \in \mathbb{Z}/r$.

Let E be a vector bundle in \mathfrak{B}_X^μ , i.e. E is a vector bundle on $X_{\mathbb{C}_p}$ of slope μ with potentially strongly semistable reduction in the sense of definition 2. We denote by r the rank of E and by d its degree, so that $\mu = d/r$. Consider a finite Galois covering $\alpha : Y \rightarrow X$ with group $G = \mu_r$ as in construction 12. We denote the unique points in Y lying over s and s_0 by t and t_0 , respectively. Let $\mathcal{O}(-dt)$ be the line bundle on Y associated to the divisor $-dt$. For all $\sigma \in G$ there is a canonical isomorphism of line bundles $\sigma^* \mathcal{O}(-dt) \rightarrow \mathcal{O}(-dt)$. Hence the line bundle $\mathcal{O}(-dt)$ carries a natural G -action. The group G_{t_0} acts trivially on the fibre of $\mathcal{O}(-dt)$ over t_0 and $G_t = G$ acts on the fibre of $\mathcal{O}(-dt)$ over t by a character $\chi : G \rightarrow \mu_r$.

The line bundle $\mathcal{O}(-dt)$ has slope $-d$. Since α has degree r , the pullback bundle $\alpha_{\mathbb{C}_p}^* E$ has slope d . By corollary 5, the bundle $F = \alpha_{\mathbb{C}_p}^* E \otimes \mathcal{O}(-dt)$ is contained in the category \mathfrak{B}_Y^0 . It is naturally equipped with isomorphisms $a_\sigma : \sigma^* F \rightarrow F$ for $\sigma \in G$ satisfying the cocycle relations. The group G_t acts by the character χ on F_t and G_{t_0} acts trivially on F_{t_0} . By theorem 9 and remark 11 the representation ρ_F of $\Pi_1(V)$ induces a continuous representation

$$\rho : \Gamma_r \rightarrow \mathrm{GL}(\rho(x_0))$$

of the group Γ_r from sequence (9) above.

We have by definition:

$$\rho(x_0) = \{(f_y) \in \prod_{\alpha(y)=x_0} (\alpha^* E)_y \otimes \mathcal{O}(-dt)_y \mid a_{\sigma,y}(f_{\sigma y}) = f_y \quad \text{for all } \sigma \in G\} .$$

Since $\mathcal{O}(-dt)$ restricted to V is the pullback of \mathcal{O}_U , we can identify $\rho(x_0)$ with E_{x_0} . Hence from ρ we obtain the desired representation

$$\rho_{E,x_0} : \Gamma_r \rightarrow \mathrm{GL}(E_{x_0}) .$$

Proposition 13 *Assuming $g \geq 1$ there is some $a \in (\mathbb{Z}/r)^*$ such that we have*

$$\rho_{E,x_0}(\zeta) = \zeta^{ad} \text{id}_{E_{x_0}} \quad \text{for all } \zeta \in \mu_r \subset \Gamma_r.$$

Proof By theorem 9 and remark 11 there is a generator τ of $G_t = G$ such that we have $\rho_{E,x_0}(\bar{\gamma}_s) = \chi(\tau)$. On the other hand $a_{\tau,t}$ acts on the fibre of $\mathcal{O}(-dt) = \mathcal{O}(-t)^{\otimes d}$ over t by multiplication with η^d where η is a primitive r 'th root of unity. Hence we have $\chi(\tau) = \eta^d$. This implies the assertion. \square

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